

CHARACTERIZATIONS OF TORIC VARIETIES VIA POLARIZED ENDOMORPHISMS

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ABSTRACT. Let X be a normal projective variety and $f : X \rightarrow X$ a polarized endomorphism. We give two characterizations for X to be a toric variety.

First we show that if X is \mathbb{Q} -factorial and G -almost homogeneous for some linear algebraic group G such that f is G -equivariant, then X is a toric variety.

Next we give a geometric characterization: if X is of Fano type and smooth in codimension 2 and if there is a reduced divisor D such that $f^{-1}(D) = D$, $f|_{X \setminus D}$ is quasi-étale and $K_X + D$ is \mathbb{Q} -Cartier, then X admits a quasi-étale cover \tilde{X} such that \tilde{X} is a toric variety and f lifts to \tilde{X} . In particular, if X is further assumed to be smooth, then X is a toric variety.

1. INTRODUCTION

We work over an algebraically closed field k of characteristic zero. Let X be a normal projective variety of dimension $n \geq 1$.

A surjective endomorphism $f : X \rightarrow X$ is *polarized*, if there is an ample Cartier divisor H such that $f^*H \sim qH$ (linear equivalence) for some integer $q > 1$. See [13] for the conjectures on polarised endomorphisms.

X is said to be *toric* or a *toric variety* if X contains an algebraic torus $T = (k^*)^n$ as an (affine) open dense subset such that the natural multiplication action of T on itself extends to an action on the whole variety X . In this case, let $D = X \setminus T$ which is a divisor; the pair (X, D) is said to be a *toric pair*.

We observe that a toric variety (e.g. the projective space) has lots of symmetries. The purpose of this short paper is to use the symmetries to characterize toric pairs.

We first give a characterization of toric varieties via polarized endomorphisms and linear group actions.

Theorem 1.1. *Let X be a normal projective variety and $f : X \rightarrow X$ a polarized endomorphism such that:*

- (i) *X is G -almost homogeneous with G being a linear algebraic group, i.e., a G -orbit U is Zariski-dense open in X ,*

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- (ii) $K_X + D$ is \mathbb{Q} -Cartier, where D is the codimension-1 part of $X \setminus U$, and
- (iii) f is G -equivariant in the sense: there is a surjective homomorphism $\varphi : G \rightarrow G$ such that $f \circ g = \varphi(g) \circ f$ for all g in G .

Then (X, D) is a toric pair.

Let (X, Δ) be a log pair. The *complexity* $c = c(X, \Delta)$ of (X, Δ) is defined as

$$c := \inf \{ n + \dim_{\mathbb{R}}(\sum \mathbb{R}[S_i]) - \sum a_i \mid \sum a_i S_i \leq \Delta, a_i \geq 0, S_i \geq 0 \};$$

see Definition 4.2. Brown, McKernan, Svaldi and Zong ([4]) recently give a geometric characterization of toric varieties involving the complexity; see [4, Theorem 1.2] or Theorem 4.3. Their result is a special case of a conjecture of Shokurov, which is stated in the relative case (cf. [12]). A simple version of their result shows that *if (X, Δ) is a log canonical pair such that Δ is reduced, $-(K_X + \Delta)$ is nef and the complexity $c(X, \Delta) \leq 0$, then (X, Δ) is a toric pair*; see Theorem 4.3 and Remark 4.4.

We show that the complexity condition of [4] holds true for the following case. A finite surjective morphism $h : Y \rightarrow Z$ between normal varieties is *quasi-étale* if h is étale outside a codimension-2 subset of Z .

Theorem 1.2. *Let X be a normal projective variety which is smooth in codimension-2, and $D \subset X$ a reduced divisor such that:*

- (i) *there is a Weil \mathbb{Q} -divisor Γ such that the pair (X, Γ) has only klt singularities;*
- (ii) *there is a polarized endomorphism $f : X \rightarrow X$ such that $f^{-1}(D) = D$ and $f|_{X \setminus D}$ is quasi-étale;*
- (iii) *the algebraic fundamental group $\pi_1^{\text{alg}}(X_{\text{reg}})$ of the smooth locus X_{reg} of X is trivial (this holds when X is smooth and rationally connected); and*
- (iv) *the irregularity $q(X) = h^1(X, \mathcal{O}_X) = 0$ (this holds when X is rationally connected).*

Then the complexity $c(X, D) \leq 0$.

Remark 1.3. Let X be an n -dimensional smooth Fano variety of Picard number one and $D \subset X$ a reduced divisor. Assume the existence of a non-isomorphic surjective endomorphism $f : X \rightarrow X$ such that $f^{-1}(D) = D$ and $f|_{X \setminus D}$ is étale. Hwang and Nakayama show that $X = \mathbb{P}^n$ and D is a simple normal crossing divisor consisting of $n + 1$ hyperplanes; see [8, Theorem 2.1]. In particular, (X, D) is a toric pair. Indeed, their argument shows that the complexity $c(X, D) \leq 0$. Our Theorem 1.2 follows their idea and tries to generalize their result to the singular case. A key step of ours is to verify that $\hat{\Omega}_X^1(\log D)$ is free, i.e., isomorphic to $\mathcal{O}_X^{\oplus n}$; see [8, Proposition 2.3] and Theorem 5.4.

As applications of Theorem 1.2, we have the following characterizations for toric pairs. A well known conjecture asserts that projective spaces are the only smooth projective Fano varieties of Picard number one admitting an endomorphism f of degree ≥ 2 (this kind of f is automatically polarized). One may generalize it to the case of arbitrary Picard number. Below is a partial solution to it.

Corollary 1.4. *Let X be a rationally connected smooth projective variety and $D \subset X$ a reduced divisor. Suppose $f : X \rightarrow X$ is a polarized endomorphism such that $f^{-1}(D) = D$ and $f|_{X \setminus D}$ is étale. Then (X, D) is a toric pair.*

The assumption below of X being of Fano type is necessary, since a normal projective toric variety is known to be of Fano type. Recall that a normal projective variety X is of *Fano type* if there is a Weil \mathbb{Q} -divisor Δ such that the pair (X, Δ) has only klt singularities and $-(K_X + \Delta)$ is an ample \mathbb{Q} -Cartier divisor.

Corollary 1.5. (cf. Remark 1.7) *Let X be of Fano type and smooth in codimension-2 and $D \subset X$ a reduced divisor. Suppose $f : X \rightarrow X$ is a polarized endomorphism such that $f^{-1}(D) = D$, $f|_{X \setminus D}$ is quasi-étale and $K_X + D$ is \mathbb{Q} -Cartier. Then there is a quasi-étale cover $\pi : \tilde{X} \rightarrow X$ and a polarized endomorphism $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$ such that:*

- (1) $\pi \circ \tilde{f} = f \circ \pi$, and
- (2) (\tilde{X}, \tilde{D}) is a toric pair, where $\tilde{D} = \pi^{-1}(D)$.

The following example satisfies the conditions of both Theorems 1.1 and 1.2.

Example 1.6. Let $X = \mathbb{P}^n$ and

$$f : X \rightarrow X; [X_0 : \cdots : X_n] \mapsto [X_0^q : \cdots : X_n^q]$$

the power map for some $q \geq 2$. Then f is polarized with $\deg f = q^n$. Let the algebraic torus $T = (k^*)^n$ act on X naturally:

$$T \times X \rightarrow X; ((t_1, \dots, t_n), [X_0 : \cdots : X_n]) \mapsto [X_0 : t_1 X_1 : \cdots : t_n X_n].$$

Let $D_i = \{X_i = 0\} \subseteq X$, $D = \sum_{i=0}^n D_i$ and $U = X \setminus D$. Let

$$\varphi : T \rightarrow T; (t_1, \dots, t_n) \mapsto (t_1^q, \dots, t_n^q)$$

which is a surjective homomorphism.

Then X is T -almost homogeneous with U the big open orbit and f is T -equivariant in the sense that $f \circ g = \varphi(g) \circ f$ for all g in T . Hence the conditions in Theorem 1.1 are all satisfied. Of course, (X, D) is a toric pair.

Note that $f^{-1}(D) = D$, $f|_{X \setminus D}$ is étale, and $\pi_1(X) = (1)$. Hence the conditions in Theorem 1.2 are all satisfied. Of course, the complexity $c(X, D) \leq n + 1 - (n + 1) = 0$.

One may take the toric blowups or blowdowns of X to get more examples satisfying all conditions in Theorems 1.1 and 1.2.

Remark 1.7. In Corollary 1.5, it is not always possible to take $\pi : \tilde{X} \rightarrow X$ to be the identity map. In other words, (X, D) itself may not be a toric pair.

Indeed, let $\tilde{f} : \tilde{X} = \mathbb{P}^n \rightarrow \tilde{X}$ be the power map of degree q^n for some $q \geq 2$ as defined in Example 1.6. The symmetric group S_{n+1} in $(n+1)$ -letters acts naturally on \mathbb{P}^n as (coordinates) permutations. Let $\tilde{D}_i := \{X_i = 0\} \subset \tilde{X}$ and $\tilde{D} = \sum \tilde{D}_i$. Then S_{n+1} fixes \tilde{D} (as a set). Choose a non-trivial subgroup $G \leq S_{n+1}$ such that:

- (i) G has no non-trivial pseudo-reflections (i.e. for any non-trivial g in G , g fixes at most a codimension-2 subset) and hence the quotient map $\pi : \tilde{X} \rightarrow X := \mathbb{P}^n/G$ is quasi-étale; and
- (ii) X has only terminal singularities and hence is smooth in codimension-2.

For instance, take $n > 2$ and $G = \langle (1, 2, \dots, n+1) \rangle \cong \mathbb{Z}/(n+1)\mathbb{Z}$; see [10, Lemma 3].

Note that $g \circ \tilde{f} = \tilde{f} \circ g$ for any g in G . Hence \tilde{f} descends to a polarized endomorphism f on X . Let $D := \pi(\tilde{D})$. Then $\pi^{-1}(D) = \tilde{D}$ and $f^{-1}(D) = D$.

We now check that f and the pair (X, D) satisfy the assumptions of Corollary 1.5. Since both $\tilde{f}|_{\tilde{X} \setminus \tilde{D}}$ and π are quasi-étale, so is $f|_{X \setminus D}$. Clearly, X is \mathbb{Q} -factorial. Since $K_{\tilde{X}} = \pi^* K_X$ is anti-ample, so is K_X . Thus, X is a Fano variety and hence of Fano type.

The pair (X, D) is not toric because the number of the irreducible components of D is less than $n+1$, G permuting the \tilde{D}_i non-trivially; see Remark 4.6. Of course, its quasi-étale cover (\tilde{X}, \tilde{D}) is a toric pair.

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2. PRELIMINARY RESULTS

2.1. Notation and terminology.

Let X be a normal projective variety of dimension $n \geq 1$. Define:

- (1) $q(X) = h^1(X, \mathcal{O}_X) = \dim H^1(X, \mathcal{O}_X)$ (the irregularity); and
- (2) $\tilde{q}(X) = q(\tilde{X})$ with \tilde{X} a smooth projective model of X .

Let D_1 and D_2 be two Cartier \mathbb{R} -divisors on X . Denote by $D_1 \equiv D_2$ if D_1 is *numerically equivalent* to D_2 , i.e., if $(D_1 - D_2) \cdot C = 0$ for every curve C on X .

Denote by X_{reg} the *smooth locus* of X . Let $U \subseteq X_{\text{reg}}$ be an open dense subset. Let $\pi : Y \rightarrow X$ be a log resolution such that π is isomorphic over U and $B = Y \setminus \pi^{-1}(U)$ is a simple normal crossing (SNC) divisor. Define the *log Kodaira dimension* of U as

$\bar{\kappa}(U) = \kappa(Y, K_Y + B)$ (Iitaka's D -dimension), which is independent of the choice of the compactification Y of U .

Given a reduced divisor D on X , we define the sheaf $\hat{\Omega}_X^1(\log D)$ as follows. Let $U \subseteq X$ be an open subset with $\text{codim}(X - U) \geq 2$ and $D \cap U$ being a normal crossing divisor. Denote by $\Omega_U^1(\log D \cap U)$ the locally free sheaf of germs of logarithmic 1-forms on U with poles only along $U \setminus D$. Using the open immersion $j : U \hookrightarrow X$, we define

$$\hat{\Omega}_X^1(\log D) = j_* \Omega_U^1(\log D \cap U).$$

This is a reflexive coherent sheaf on X .

Throughout this paper, for a pair (X, Δ) , the coefficients of Δ lie in $[0, 1]$.

The result below is frequently used and part of [11, Proposition 2.5].

Lemma 2.2. *Let $f : X \rightarrow X$ be a polarized endomorphism of a normal projective variety X . Suppose $f^*M \equiv M$ (numerical equivalence) for some Cartier \mathbb{R} -divisor M . Then $M \equiv 0$.*

3. PROOF OF THEOREM 1.1

Lemma 3.1. *Let X be a normal projective variety with an algebraic torus T -action. Suppose T has a Zariski-dense open orbit U in X . Then X is a toric variety.*

Proof. Let $x \in U$. Then $U = T/T_x$, where $T_x := \{t \in T \mid tx = x\}$. Since T is a torus, $U = T/T_x$ is again a torus. For any $y \in U$ and $t \in T_x$, $y = t_y x$ for some $t_y \in T$ and hence $ty = tt_y x = t_y tx = y$. In particular, T_x acts trivially on U and hence on X . So the natural action of $U = T/T_x$ on itself may extend to X . So X is a toric variety. \square

Lemma 3.2. *Let X be a normal projective variety and $U \subseteq X$ a Zariski-dense open subset such that U is contained in the smooth locus of X . Let D be the sum of all the prime divisors D_i contained in $X \setminus U$. Assume (X, D) is log canonical. Let $\pi : Y \rightarrow X$ be a resolution such that π is isomorphic over U and $Y \setminus \pi^{-1}(U)$ is an SNC divisor. Let D' be the strict transform of D and E the sum of π -exceptional prime divisors E_j such that $Y \setminus \pi^{-1}(U) = D' + E$. Then the log Kodaira dimension $\bar{\kappa}(U) = \kappa(Y, D' + E)$ equals $\kappa(X, K_X + D)$, and $K_Y + D' + E$ is pseudo-effective (i.e., the limit of effective divisors) if and only if so is $K_X + D$.*

Proof. Since (X, D) is log canonical, $K_Y + D' + E = \pi^*(K_X + D) + \sum_j b_j E_j$ with $b_j \geq 0$. Hence the lemma follows by the projection formula. \square

Proof of Theorem 1.1. Write $\deg f = q^{\dim(X)}$ with $q > 1$. We may assume G is connected and acts faithfully on X . Let $U = Gx_0$ be the open dense G -orbit in X . Note that

$f(U) = f(Gx_0) = \varphi(G)f(x_0) = Gf(x_0)$ is an orbit of dimension equal to $\dim U = \dim X$, and hence $f(U) = U$. So $f^{-1}(U) \supseteq U$. Further, we claim that $f^{-1}(U) = U$. Indeed, for any $x \notin U$, the orbit Gx has $\dim(Gx) < \dim(X)$. Hence $Gf(x) = \varphi(G)f(x) = f(Gx)$ has dimension equal to $\dim(Gx)$ ($< \dim(X)$). Thus $f(x)$ is not in U . The claim is proved. Hence $f^{-1}(X \setminus U) = X \setminus U$. So $f^{-1}(D) = D$ where D is the divisorial part of $X \setminus U$.

Since U is G -transitive, $f|_U : f^{-1}(U) = U \rightarrow U$ is étale. Thus, by the logarithmic ramification divisor formula, we have $K_X + D = f^*(K_X + D)$. Hence, $K_X + D \equiv 0$ by Lemma 2.2. Since $K_X + D$ is \mathbb{Q} -Cartier, (X, D) is log canonical by [3, Theorem 1.4]. Then, in notation of Lemma 3.2, the log canonical divisor $K_Y + D' + E$ of the smooth compactification $(Y, D' + E)$ of U is pseudo-effective. Fix some u in U . Let $H := \{g \in G \mid gu = u\}$. Then $U = G/H$. Since G is linear and acts faithfully on X , G is an algebraic torus by [2, Theorem 1.1]. So X is a toric variety by Lemma 3.1. Since the big torus U is an affine variety, $X \setminus U$ is of pure co-dimension one and hence equal to D . Thus (X, D) is a toric pair. \square

4. THE COMPLEXITY

4.1. Some notations. Let X be an n -dimensional normal projective variety and $D = \sum_{i=1}^{\ell} D_i$ a reduced divisor on X . Denote by $\ell(D) := \ell$, the *number* of irreducible components in D ; and $r(D)$ the *rank* of the vector space spanned by D_1, \dots, D_{ℓ} in the space of Weil \mathbb{R} -divisors modulo algebraic equivalence.

Let (X, Δ) be a log pair. Write $\Delta = \sum_i a_i D_i$ with D_i distinct irreducible divisors. Denote by

$$\langle \Delta \rangle := \lfloor \Delta \rfloor + \lceil 2\Delta \rceil - \lfloor 2\Delta \rfloor = \sum_{i: a_i > 1/2} D_i.$$

Definition 4.2. A *decomposition* of Δ is an expression of the form

$$\sum a_i S_i \leq \Delta,$$

where $S_i \geq 0$ are \mathbb{Z} -divisors and $a_i \geq 0$, $1 \leq i \leq k$. The *complexity* of this decomposition is $n + r - d$, where r is the rank of the vector space spanned by S_1, S_2, \dots, S_k in the space of Weil \mathbb{R} -divisors modulo algebraic equivalence and $d = \sum a_i$. The *complexity* $c = c(X, \Delta)$ of (X, Δ) is the infimum of the complexity of any decomposition of Δ .

The following theorem gives a geometric characterization of toric varieties involving the complexity by Brown, McKernan, Svaldi and Zong; see [4, Theorem 1.2].

Theorem 4.3. (cf. [4]) *Let X be a proper variety of dimension n and let (X, Δ) be a log canonical pair such that $-(K_X + \Delta)$ is nef. If $\sum a_i S_i$ is a decomposition of complexity c*

less than one then there is a divisor D such that (X, D) is a toric pair, where $D \geq \langle \Delta \rangle$ and all but $\lfloor 2c \rfloor$ components of D are elements of the set $\{S_i \mid 1 \leq i \leq k\}$.

Remark 4.4. (1) If the Δ in Theorem 4.3 is a reduced divisor with the complexity $n + r(\Delta) - \ell(\Delta) \leq 0$, then Theorem 4.3 implies that (X, Δ) itself is a toric pair.

(2) Let $f : X \rightarrow X$ be a polarized endomorphism and D is a reduced divisor such that $f^{-1}(D) = D$, $f|_{X \setminus D}$ is quasi-étale and $K_X + D$ is \mathbb{Q} -Cartier. Then $K_X + D \equiv 0$ by the logarithmic ramification divisor formula and Lemma 2.2. In particular, $-(K_X + D)$ is nef; further, (X, D) is log canonical (cf. [3, Theorem 1.4]).

The following theorem provides us with a useful upper bound of the complexity.

Theorem 4.5. *Let X be a normal projective variety and D a reduced divisor of X . Then, in notations of 2.1 and 4.1, we have*

$$\ell(D) \geq h^0(X, \hat{\Omega}_X^1(\log D)) + r(D) - \tilde{q}(X).$$

In particular, the complexity

$$c(X, D) \leq n + \tilde{q}(X) - h^0(X, \hat{\Omega}_X^1(\log D)).$$

Proof. Let $\pi : \tilde{X} \rightarrow X$ be a log resolution of the pair (X, D) . Set $\tilde{D} :=$ largest reduced divisor contained in $\text{Supp } \pi^{-1}(\text{non-klt locus of } (X, D))$. Write $\tilde{D} = \sum \tilde{D}_i = D' + E$, where the \tilde{D}_i are the irreducible components of \tilde{D} , D' is the strict transform of D , and E is the π -exceptional part.

From the exact sequence

$$0 \rightarrow \Omega_{\tilde{X}}^1 \rightarrow \Omega_{\tilde{X}}^1(\log \tilde{D}) \rightarrow \oplus_i \mathcal{O}_{\tilde{D}_i} \rightarrow 0,$$

we get

$$0 \rightarrow H^0(\tilde{X}, \Omega_{\tilde{X}}^1) \rightarrow H^0(\tilde{X}, \Omega_{\tilde{X}}^1(\log \tilde{D})) \rightarrow \oplus_i H^0(\tilde{D}_i, \mathcal{O}_{\tilde{D}_i}) \rightarrow H^1(\tilde{X}, \Omega_{\tilde{X}}^1),$$

where the connecting homomorphism essentially sends a generator 1 of $\mathcal{O}_{\tilde{D}_i}$ for each component \tilde{D}_i to the first Chern class $c_1(\tilde{D}_i)$. So $\ell(\tilde{D}) = h^0(\tilde{X}, \Omega_{\tilde{X}}^1(\log \tilde{D})) + r(\tilde{D}) - q(\tilde{X})$.

By [6, Theorem 1.5], $\mathcal{F} := \pi_* \Omega_{\tilde{X}}^1(\log \tilde{D})$ is reflexive. Note that $\mathcal{F}|_U = \Omega_U^1(\log D \cap U)$, where $U \subseteq X_{\text{reg}}$ such that $\text{codim}(X \setminus U) \geq 2$ and $D \cap U$ is a smooth divisor. Then $H^0(\tilde{X}, \Omega_{\tilde{X}}^1(\log \tilde{D})) = H^0(X, \mathcal{F}) = H^0(U, \mathcal{F}|_U) = H^0(X, \hat{\Omega}_X^1(\log D))$.

By the negativity lemma (cf. [1, Lemma 3.6.2]), we have $r(E) = \ell(E)$ and hence $r(D) \leq r(\tilde{D}) - \ell(E)$. So the Lemma is proved. \square

Remark 4.6. In Theorem 4.5, if X is assumed to be \mathbb{Q} -factorial, then the negativity lemma implies $r(D) = r(\tilde{D}) - \ell(E)$ at the end of the proof. In particular, we will have $\ell(D) = h^0(X, \hat{\Omega}_X^1(\log D)) + r(D) - \tilde{q}(X)$ and $c(X, D) = n + \tilde{q}(X) - h^0(X, \hat{\Omega}_X^1(\log D))$.

If (X, D) is assumed to be a normal projective toric pair, then it is known that $\hat{\Omega}_X^1(\log D)$ is free; see [5, 4.3, page 87]. Since X is rationally connected, $\tilde{q}(X) = 0$. Therefore, $\ell(D) \geq n + r(D)$ (with equality holding true when X is \mathbb{Q} -factorial).

5. PROOF OF THEOREM 1.2

Lemma 5.1. *Let X be a normal projective variety with finite algebraic fundamental group $\pi_1^{\text{alg}}(X_{\text{reg}})$. Then X admits a universal quasi-étale cover $\pi : \tilde{X} \rightarrow X$, such that $\pi_1^{\text{alg}}(\tilde{X}_{\text{reg}})$ is trivial and any surjective endomorphism f of X lifts to \tilde{X} .*

Proof. Since $\pi_1^{\text{alg}}(X_{\text{reg}})$ is finite, there is a universal quasi-étale cover $\pi : \tilde{X} \rightarrow X$ such that $\pi_1^{\text{alg}}(\tilde{X}_{\text{reg}})$ is trivial. Let W be the normalization of the fibre product of f and π and W_0 a dominant irreducible component of W . Then $W_0 \rightarrow X$ is also a quasi-étale cover. Taking the universal quasi-étale cover of W_0 which is \tilde{X} , we are done. \square

The same argument of [8, Proposition 2.4] gives the following.

Proposition 5.2. *Let X be a normal projective variety smooth in codimension-2 and $D \subset X$ a reduced divisor. Suppose $f : X \rightarrow X$ is a polarized endomorphism such that $f^{-1}(D) = D$ and $f|_{X \setminus D}$ is quasi-étale. Then there is a smooth open subset $U \subseteq X$ such that $D \cap U$ is a normal crossing divisor and $\text{codim}(X \setminus U) \geq 3$. In particular, $\hat{\Omega}_X^1(\log D)$ is locally free over U .*

The following slightly extends [8, Propositions 2.2 and 2.3].

Proposition 5.3. *Let X be a normal projective variety which is of dimension $n \geq 2$ and smooth in codimension-2, and $D \subset X$ a reduced divisor. Suppose $f : X \rightarrow X$ is a polarized endomorphism such that $f^{-1}(D) = D$ and $f|_{X \setminus D}$ is quasi-étale. Let H be an ample divisor such that $f^*H \sim qH$ for some $q > 1$. Then the following hold.*

- (1) $c_1(\hat{\Omega}_X^1(\log D)) \cdot H^{n-1} = c_1(\hat{\Omega}_X^1(\log D))^2 \cdot H^{n-2} = c_2(\hat{\Omega}_X^1(\log D)) \cdot H^{n-2} = 0$.
- (2) $\hat{\Omega}_X^1(\log D)$ is an H -slope semistable reflexive sheaf.

Proof. By Proposition 5.2, there is a smooth open subset $U \subseteq X$ such that $D \cap U$ is a normal crossing divisor and $\text{codim}(X \setminus U) \geq 3$. Since $f|_{X \setminus D}$ is quasi-étale, $f|_{f^{-1}(U) \setminus D}$ is étale by the purity of branch loci.

There is a natural morphism $\varphi : f^*\hat{\Omega}_X^1(\log D) \rightarrow \hat{\Omega}_X^1(\log D)$ and $\varphi|_{f^{-1}(U)}$ is an isomorphism. So for $i = 1, 2$, $f^*c_i(\hat{\Omega}_X^1(\log D)) = c_i(f^*\hat{\Omega}_X^1(\log D)) = c_i(\hat{\Omega}_X^1(\log D))$. Then

$$q^{n-i}c_i(\hat{\Omega}_X^1(\log D)) \cdot H^{n-i} = f^*c_i(\hat{\Omega}_X^1(\log D)) \cdot (f^*H)^{n-i} = (\deg f)c_i(\hat{\Omega}_X^1(\log D)) \cdot H^{n-i}$$

implies $c_i(\hat{\Omega}_X^1(\log D)) \cdot H^{n-i} = 0$ for $i = 1, 2$, noting that $\deg f = q^n > 1$. The proof for $c_1(\hat{\Omega}_X^1(\log D))^2 \cdot H^{n-2} = 0$, is similar.

For (2), suppose the contrary that $\hat{\Omega}_X^1(\log D)$ is not H -slope semistable. Then there is a coherent subsheaf $\mathcal{F} \subset \hat{\Omega}_X^1(\log D)$ such that

$$\mu_H(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\text{rank } \mathcal{F}} > 0.$$

Note that

$$s = \sup\{\mu_H(\mathcal{F}) \mid \mathcal{F} \subset \hat{\Omega}_X^1(\log D)\} < \infty.$$

So for some $k \gg 1$, $\mu_H(g^*\mathcal{F}) = q^k \mu_H(\mathcal{F}) > s$ with $g = f^k$. Let $i : f^{-1}(U) \hookrightarrow X$ be the inclusion map and let $\mathcal{G} := i_*((g^*\mathcal{F})|_{g^{-1}(U)})$. Then $\mu_H(\mathcal{G}) = \mu_H(g^*\mathcal{F}) > s$. Note that $(g^*\mathcal{F})|_{g^{-1}(U)}$ is a subsheaf of the locally free sheaf $(g^*\hat{\Omega}_X^1(\log D))|_{g^{-1}(U)} \cong \hat{\Omega}_X^1(\log D)|_{g^{-1}(U)}$. Since i_* is left exact and $\text{codim}(X \setminus g^{-1}(U)) \geq 3$, \mathcal{G} is a coherent subsheaf of the reflexive sheaf $\hat{\Omega}_X^1(\log D)$. So we get a contradiction and (2) is proved. \square

Theorem 5.4. *Let X be a normal projective variety which is smooth in codimension-2, and $D \subset X$ a reduced divisor such that:*

- (i) *there is a Weil \mathbb{Q} -divisor Γ such that the pair (X, Γ) has only klt singularities;*
- (ii) *$f : X \rightarrow X$ is a polarized endomorphism such that $f^{-1}(D) = D$ and $f|_{X \setminus D}$ is quasi-étale; and*
- (iii) *the algebraic fundamental group $\pi_1^{\text{alg}}(X_{\text{reg}})$ of the smooth locus X_{reg} of X is trivial.*

Then $\hat{\Omega}_X^1(\log D)$ is free, i.e., isomorphic to $\mathcal{O}_X^{\oplus n}$, where $n := \dim X$.

Proof. The case $n = 1$ is clear. If $n \geq 2$, apply Proposition 5.3 and [7, Theorem 1.20]. \square

Proof of Theorem 1.2. If $\dim(X) = 1$, then $X \cong \mathbb{P}^1$ since $\pi_1^{\text{alg}}(X_{\text{reg}})$ is trivial and $\deg f > 1$. Clearly, $\ell(D) = \deg D = 2$ and $r(D) = 1$; see Remark 4.4. So $c(X, D) = 0$. Now we may assume $\dim(X) > 1$. By Theorem 5.4, $\hat{\Omega}_X^1(\log D)$ is free. Since (X, Γ) is klt, X has rational singularities. So we have $\tilde{q}(X) = q(X) = 0$ by the assumption. Thus $c(X, D) \leq \tilde{q}(X) = 0$ (cf. Theorem 4.5). \square

Proof of Corollary 1.4. By Theorem 1.2, $c(X, D) \leq 0$. So (X, D) is a toric pair by Remark 4.4 and [4, Theorem 1.2] or Theorem 4.3. \square

Proof of Corollary 1.5. Since X is of Fano type, there is a Weil \mathbb{Q} -divisor Δ such that the pair (X, Δ) is klt and $-(K_X + \Delta)$ is ample. By [7, Theorem 1.13], $\pi_1^{\text{alg}}(X_{\text{reg}})$ is finite. By Lemma 5.1, X admits a universal quasi-étale cover $\pi : \tilde{X} \rightarrow X$, such that $\pi_1^{\text{alg}}(\tilde{X}_{\text{reg}})$ is trivial and f lifts to \tilde{f} on \tilde{X} . Note that \tilde{X} , like X , is still smooth in codimension-2 by the purity of branch loci.

Let $\tilde{D} := \pi^{-1}(D)$ and $\tilde{\Delta} = \pi^*(\Delta)$. Then $K_{\tilde{X}} + \tilde{D}$ equals $\pi^*(K_X + D)$ and hence is \mathbb{Q} -Cartier (and numerically trivial; see Remark 4.4). Also $K_{\tilde{X}} + \tilde{\Delta}$ equals $\pi^*(K_X + \Delta)$ and is hence anti-ample. Note that $(\tilde{X}, \tilde{\Delta})$ is also klt (cf. [9, Proposition 5.20]). Hence

\tilde{X} is also of Fano type, so $q(\tilde{X}) = 0$ by the Kawamata-Viehweg vanishing. Since \tilde{f} is the lifting of f , it is polarized and $\tilde{f}^{-1}(\tilde{D}) = \tilde{D}$. Since both $\pi : \tilde{X} \rightarrow X$ and $f|_{X \setminus D}$ are quasi-étale, so is $\tilde{f}|_{\tilde{X} \setminus \tilde{D}}$.

Thus Theorem 1.2 is applicable: $c(\tilde{X}, \tilde{D}) \leq 0$. So (\tilde{X}, \tilde{D}) is a toric pair by Remark 4.4 and [4, Theorem 1.2] or Theorem 4.3. \square

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